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Stability analysis of prey-predator model with alternative food sources and transition two diseases in the same population

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Abstract- In this paper, the effect of alternative food sources and transitive two different types of diseases in the ecological models, specifically a prey-predator model, is proposed and studied. Both of the diseases transition in the same population, specifically in the predators. The first one of which the SIS-epidemics is transmitted. The second one of which the SI-epidemics is transmitted. The model is characterized by a four of autonomous nonlinear differential equations with nonnegative parameters. All the model's equilibriums are determined and the dynamic behaviors of the model near them are investigated. Finally, contains the numerical simulation investigation at each equilibrium points

Keywords- prey-predator model, SI epidemics disease, SIS epidemics disease, Lyapunov function, boundedness, stability analysis.

1. INTRODUCTION:

Diseases in a prey-predator system have received significant interest in resent years. It is well known that, in nature species does not exist alone. In fact, any given habitat may contain dozens or hundreds of species, some times thousands. Since any species has at least the potential to interact with any other species in its habitat, the possibility of spread of the disease in a community rapidly becomes astronomical as the number of infected species in the habitat increases. Therefore, it is more of biological significance to study the effect of disease on the dynamical behavior of interacting species

many researchers, especially in the last two decades, have proposed and studied different prey-predator models in the presence of disease in one of the species see for example [1-13] and the references there in. In most previous studies, the only means of transmission of disease is the direct contact between individuals. However, many diseases are transmitted in the species not only through contact, but also directly from environment.

Elisa Elena et al [14] proposed prey-predator model two diseases affect the prey. Predators are allowed to have other food sources. Fabio Roman et al [15] proposed prey-predator model containing two disease strains in the predator population.

On contrast to all of the above studies, in this paper a prey-predator model involving SIS and SI infectious diseases in predator species is proposed and analyzed. It is assumed that the predator population has external source of food. It is assumed that both of the diseases spread within predator population by contact between susceptible individuals and infected individuals. Further, in this model, linear type of functional response as well as linear incidence rate for describing the transition both of disease are used.

2. MATHEMATICAL MODEL:

The basic prey-predator model is

$$\frac{dP}{dT} = P(a - bP) - cPN$$
$$\frac{dN}{dT} = N(ecP - \theta) \tag{1}$$

where $P(T)$ and $N(T)$ represent the densities of prey and predator species at time T respectively. Clearly the above model is a simple Lotka-Volterra prey-predator model with logistic growth rate for prey. The positive parameters a, b, c, e and θ represent intrinsic growth rate, intra-specific competition, attack rate, conversion rate and natural death rate respectively [16].

We impose the following assumptions:

2.1 In the presence of first disease, SIS disease, the predator population consists of two subclasses, namely, the susceptible predator $S(T)$ and the infected predator by this disease $I_1(T)$.

2.2 In the presence of second disease, SI disease, the predator population consists of two subclasses, namely, the susceptible predator $S(T)$ and the infected predator by this disease $I_2(T)$. Therefore at any time T we have $N(T) = S(T) + I_1(T) + I_2(T)$.

2.3 The susceptible predator has an alternative food sources supplied by a constant rate $\beta > 0$.

2.4 Both of the diseases, SIS and SI, transmitted among the predator individuals only, but not the prey individuals, by contact with an infected predator at infection rate $\alpha_1 > 0$ and $\alpha_2 > 0$ respectively.

2.5 Only the first disease disappears and the infected predator becomes susceptible predator again at a recover rate $w > 0$. Finally both of the diseases, SIS and SI, induces the mortality within the infected predator individuals at a constant rate $\delta_1 > 0$ and $\delta_2 > 0$.

2.6 The infected predator, by SIS disease, feed on the prey species according to Lotka-Volterra functional response with attack rate constant $\tau_1 > 0$. Also, the infected predator by SI disease feed on the prey species by functional response with attack rate constant $\tau_2 > 0$.

These assumptions can be mathematically realized into the following four differential equations

$$\begin{aligned}
\frac{dP}{dT} &= P[(a-bP)-cS-c\tau_1 I_1 - c\tau_2 I_2] \\
\frac{dS}{dT} &= S(ecP-\alpha_1 I_1-\alpha_2 I_2-\theta+\beta)+wI_1 \\
\frac{dI_1}{dT} &= I_1(ec\tau_1 P+\alpha_1 S-\theta-\delta 1-w) \\
\frac{dI_2}{dT} &= I_2(ec\tau_2 P+\alpha_2 S-\theta-\delta_2)
\end{aligned} \qquad (2)$$

In order to simplifying the proposed model (2), the following dimensionless variables are used:

$$t=aT,\ p=\frac{c}{a}P,\ \ s=\frac{c}{a}S,\ \ y_1=\frac{c}{a}I_1,\ \ y_2=\frac{c}{a}I_2$$

Thus we obtain the following dimensionless form of the model (3):

$$\begin{aligned}
\frac{dp}{dt} &= p[(1-h_1 p)-s-\tau_1 y_1-\tau_2 y_2] \\
\frac{ds}{dt} &= s(ep-h_2 y_1-h_3 y_2+h_4)+h_5 y_1 \\
\frac{dy_1}{dt} &= y_1(e\tau_1 p+h_2 s-h_5-h_6) \\
\frac{dy_2}{dt} &= y_2(e\tau_2 p+h_3 s-h_7)
\end{aligned} \qquad (3)$$

Where:

$$h_1=\frac{b}{a}>0,\ h_2=\frac{\alpha_1}{c}>0,\ h_3=\frac{\alpha_2}{c}>0,\ h_4=\frac{\beta-\theta}{a}\in\Re,$$
$$h_5=\frac{w}{c}>0,\ h_6=\frac{\theta+\delta_1}{a}>0,\ h_7=\frac{\theta+\delta_2}{a}>0$$

represent the dimensionless parameters of the model (2). The initial condition for model (3) may be taken as any point in the region $\Re_+^4$. Obviously, the interaction functions in the right hand side of system (3) are continuously differentiable functions on $\Re_+^4$, hence they are Lipschitizian. Therefore the solution of system (3) exists and is unique. Further, all the solutions of system (3) with non-negative initial condition are uniformly bounded as shown in the following theorem.

THEOREM (1): All the solutions of system (3), which initiate in $\Re_+^4$ are uniformly bounded if the sufficient condition $h_4<0$ holds.

PROOF: From the first equation of system (3) we obtain that;

$$\frac{dp}{dt}\le p(1-h_1 p)$$

Clearly by solving the above differential inequality we get

$$\lim_{t\to\infty}\sup p(t)\le\frac{1}{h_1}$$

Define the function $\mathrm{M}(t)=p(t)+\frac{1}{e}s(t)+\frac{1}{e}y_1(t)+\frac{1}{e}y_2(t)$ and then take its time derivative along the solution of system (3), gives

$$\frac{d\mathrm{M}}{dt}\le p-\frac{\phi}{e}s-\frac{\phi}{e}y_1-\frac{\phi}{e}y_2 \quad \text{where } \phi=\min\{-h_4,h_6,h_7\}$$
$$\le \pi-\phi\mathrm{M} \qquad \text{where } \pi=(1+\phi)\frac{1}{H}$$

Now, by using Gronwall lemma [17], it obtains that:

$$0<\mathrm{M}(t)\le\mathrm{M}(0)e^{-\phi t}+\frac{\pi}{\phi}\left(1-e^{-\phi t}\right)$$

which yields $\lim_{t\to\infty}\sup\mathrm{M}(t)\le\frac{\pi}{\phi}$ that is independent of the initial conditions. ■

3. EXISTENCE OF EQUILIBRIUM POINTS:

The system (3) has at most eleven biologically feasible equilibrium points, namely $E_k=(p_k,s_k,y_{1k},y_{2k}),\ k=0,1,2,...,10$. The existence conditions for each of these equilibrium points are discussed in the following:

3.1 The ***vanishing equilibrium point*** $E_0=(0,0,0,0)$ always exists.

3.2 The ***axial equilibrium point on the*** s ***-axis*** $E_1=(0,s_1,0,0)$ where s_1 is any positive number, exists if and only if $h_4=0$.

3.3 The ***axial equilibrium point on the*** p ***-axis*** $E_2=(p_2,0,0,0)$ where $p_2=1/h_1$, always exists.

3.4 The ***first disease and prey free equilibrium point*** $E_3=(0,s_3,0,y_{23})$ where:

$$s_3=\frac{h_7}{h_3} \text{ and } y_{23}=\frac{h_4}{h_3} \qquad (4)$$

exists uniquely in the interior of the first quadrant of sy_2-plane under the following necessary and sufficient condition $h_4>0$.

3.5 The ***second disease and prey free equilibrium point*** $E_4=(0,s_4,y_{14},0)$ where:

$$s_4=\frac{h_5+h_6}{h_2} \text{ and } y_{14}=\frac{h_4(h_5+h_6)}{h_2 h_6} \qquad (5)$$

exists uniquely in the interior of the first quadrant of sy_1-plane under the following necessary and sufficient condition $h_4>0$.

3.6 The ***first disease and susceptible predator free equilibrium point*** $E_5=(p_5,0,0,y_{25})$ where:

$$p_5=\frac{h_7}{e\tau_2} \text{ and } y_{25}=\frac{e\tau_2-h_1 h_7}{e\tau_2^2} \qquad (6)$$

exists uniquely in the interior of the first quadrant of py_2-plane under the following necessary and sufficient condition $e\tau_2>h_1 h_7$.

3.7 The ***disease free equilibrium point*** $E_6=(p_6,s_6,0,0)$ where:

$$p_6=\frac{-h_4}{e} \text{ and } s_6=1-h_1 p_6 \qquad (7)$$

exists uniquely in the interior of the first quadrant of $ps-$plane under the following necessary and sufficient conditions $h_4<0$ and $1>h_1 p_6$.

3.8 The ***second disease free equilibrium point*** $E_7=(p_7,s_7,y_{17},0)$ where:

$$s_7=\frac{h_5+h_6-e\tau_1 p_7}{h_2} \text{ and } y_{17}=\frac{(h_5+h_6-e\tau_1 p_7)(ep_7+h_4)}{h_2(h_6-e\tau_1 p_7)} \qquad (8)$$

while p_7 represents a positive root of the following second order polynomial equation

$A_1p^2 + A_2p + A_3 = 0$

where

$A_1 = e\tau_1 h_1 h_2 > 0$;

$A_2 = -\left(h_1h_2h_6 + e\tau_1 h_2 - e\tau_1\left(h_6 + \tau_1 h_4\right)\right)$;

$A_3 = h_2h_6 - \left(h_5 + h_6\right)\left(h_6 + \tau_1 h_4\right)$;

Therefore, straight forward computation shows that E_7 exists uniquely in the interior of the first octant of psy_1 – plane if and only if the following conditions are hold.

$ep_7 > -h_4$, $h_6 > \max\{e\tau_1 p_7, \tau_1 h_4\}$ and

$h_2h_6 < \left(h_5 + h_6\right)\left(h_6 + \tau_1 h_4\right)$

3.9 The ***first disease free equilibrium point*** $E_8 = \left(p_8, s_8, 0, y_{28}\right)$ where:

$$p_8 = \frac{h_3 - h_7 - \tau_2 h_4}{h_1 h_3}, \quad s_8 = \frac{h_7 - e\tau_2 p_8}{h_3} \text{ and } y_{28} = \frac{ep_8 + h_4}{h_3} \tag{9}$$

exists uniquely in the interior of the first octant of psy_2 – plane under the following necessary and sufficient conditions $h_4 > 0$, $p_8 < \dfrac{h_7}{e\tau_2}$ and $h_3 > h_7 + \tau_2 h_4$.

3.10 The ***prey free equilibrium point*** $E_9 = \left(0, s_9, y_{19}, y_{29}\right)$ where:

$s_9 = \dfrac{h_5 + h_6}{h_2} = \dfrac{h_7}{h_3}$ and $y_{19} = \dfrac{\left(h_5 + h_6\right)\left(h_4 - h_3 y_{29}\right)}{h_2 h_6}$ where y_{29} is any positive number, E_9 exists uniquely in the interior of the first octant of sy_1y_2 – plane under the following necessary and sufficient conditions $h_3\left(h_5 + h_6\right) = h_2 h_7$, $h_4 > 0$ and $h_4 > h_3 y_{29}$.

3.11 The coexistence equilibrium point $E_{10} = \left(p_{10}, s_{10}, y_{110}, y_{210}\right)$ where

$$p_{10} = \frac{h_2h_7 - h_3\left(h_5 + h_6\right)}{e\left(\tau_2 h_2 - \tau_1 h_3\right)}; \quad s_{10} = \frac{\tau_2\left(h_5 + h_6\right) - \tau_1 h_7}{\left(\tau_2 h_2 - \tau_1 h_3\right)};$$

$$y_{210} = \frac{1}{\tau_2}\left[1 - h_1 p_{10} - s_{10} - \tau_1 y_{110}\right]$$

$$y_{110} = \frac{s_{10}\left[e\left(\tau_2 h_2 + \tau_1 h_3\right)\left(h_7 + \tau_2 h_4 - h_3\right) + h_1 h_3\left(h_2 h_7 - h_3\left(h_5 + h_6\right)\right)\right]}{e\left(\tau_2 h_6 - \tau_1 h_7\right)\left(\tau_2 h_2 - \tau_1 h_3\right)}$$

Therefore, straight forward computation shows that E_{10} exists uniquely in the Int. $\Re^4_+$ if and only if the following conditions are hold.

$\max\left\{\dfrac{h_3\left(h_5 + h_6\right)}{h_2}, h_3 - \tau_2 h_4\right\} < h_7 < \dfrac{\tau_2 h_6}{\tau_1}$; $\tau_2 h_2 > \tau_1 h_3$ and

$1 > h_1 p_{10} + s_{10} + \tau_1 y_{110}$

The Jacobian matrix of system (3) is $J = (\beta_{ij}) \in \Re_{4x4}$, with entries

$\beta_{11} = 1 - 2h_1 p - s - \tau_1 y_1 - \tau_2 y_2$, $\beta_{12} = -p$, $\beta_{13} = -\tau_1 p$, $\beta_{14} = -\tau_2 p$, $\beta_{21} = es$, $\beta_{22} = ep - h_2 y_1 - h_3 y_2 + h_4$, $\beta_{23} = -h_2 s + h_5$, $\beta_{24} = -h_3 s$, $\beta_{31} = e\tau_1 y_1$, $\beta_{32} = h_2 y_1$, $\beta_{33} = e\tau_1 p + h_2 s - h_5 - h_6$, $\beta_{34} = 0$, $\beta_{41} = e\tau_2 y_2$, $\beta_{42} = h_3 y_2$, $\beta_{43} = 0$, $\beta_{44} = e\tau_2 p + h_3 s - h_7$

In what follows, the system's equilibria are E_k and we denote by J_k and $\beta^{[k]}_{ij}$ the Jacobian and its entries evaluated at E_k, $i = 1,\ldots,4$, $j = 1,\ldots,4$, $k = 0,1,2,\ldots,10$

4. The Stability Analysis:

The equilibria E_0 is saddle point, since its eigenvalues are $1 > 0$, h_4, $-\left(h_5 + h_6\right) < 0$ and $-h_7 < 0$.

Theorem (2): The non-hyperbolic equilibrium point E_1 is locally asymptotically stable in $\Re^4_+$ if and only if:

$$1 - h_1 p < s_1 < \min\left\{s, \frac{h_7}{h_3}, \frac{h_6 s}{h_2 s - h_5}\right\} \tag{10}$$

Proof: Consider the function

$$V^{[1]} = p + \frac{1}{e}\left(s - s_1 - s_1 \ln\frac{s}{s_1}\right) + \frac{y_1}{e} + \frac{y_2}{e}$$

Clearly, $V^{[1]}: \Re^4_+ \to \Re$ and $V^{[1]}(E_1) = 0$ with $V^{[1]}(E) \neq 0 \ \ \forall E \neq E_1, E \in \Re^4_+$. Hence it is positive definite function in $\Re^4_+$. Also, the derivative of $V^{[1]}$ with respect to the time t is given as follows.

$$\frac{dV^{[1]}}{dt} = p\left(1 - h_1 p - s_1\right) + \frac{y_1}{e}\left(h_2 s_1 - \frac{h_5}{s}s_1 - h_6\right) + \frac{h_4}{e}\left(s - s_1\right) + \frac{y_2}{e}\left(h_3 s_1 - h_7\right)$$

Since E_1 exists if and only if $h_4 = 0$, in addition condition (10), guarantee that $\dfrac{dV^{[1]}}{dt} < 0$ on subregion of $\Re^4_+$, then $V^{[1]}$ is a Lyapunov function on that subregion which satisfy condition (10). since $\dfrac{dV^{[1]}}{dt} < 0$ on subregion of $\Re^4_+$ then E_1 is a locally asymptotically stable but not globally. ■

Theorem (3): The equilibrium point E_2 is locally asymptotically stable in $\Re^4_+$ if and only if:

$$ep_2 < \min\left\{-h_4, \frac{h_5 + h_6}{\tau_1}, \frac{h_7}{\tau_2}\right\} \tag{11}$$

Proof: The Jacobian matrix of the system (3) at E_2 is given by:

$$J_2 = \begin{pmatrix} 1 - 2h_1 p_2 & -p_2 & -\tau_1 p_2 & -\tau_2 p_2 \\ 0 & ep_2 + h_4 & h_5 & 0 \\ 0 & 0 & e\tau_1 p_2 - h_5 - h_6 & 0 \\ 0 & 0 & 0 & e\tau_2 p_2 - h_7 \end{pmatrix}$$

So, the characteristic equation of J_2 can be written by

$$\left(1 - 2h_1 p_2 - \mu_p\right)\left(ep_2 + h_4 - \mu_s\right)\left(e\tau_1 p_2 - h_5 - h_6 - \mu_{y1}\right)\times \left(e\tau_2 p_2 - h_7 - \mu_{y_2}\right) = 0$$

from which, we obtain that:

$\mu_p = 1 - 2h_1 p_2 < 0$, $\mu_s = e p_2 + h_4$, $\mu_{y1} = e\tau_1 p_2 - h_5 - h_6$ and $\mu_{y2} = e\tau_2 p_2 - h_7$

Here μ_p, μ_s, μ_{y1} and μ_{y2} denote to the eigenvalues in the $p-$direction, $s-$direction, y_1-direction and y_2-direction, respectively. So, it is easy to verify that, all the eigenvalues have negative real parts if and only if the condition (11) holds. Therefore, the equilibrium point E_2 is locally asymptotically stable in $\Re_+^4$. Furthermore, it is a globally asymptotically stable too. ■

THEOREM (4): the non-hyperbolic equilibrium point $E_3 = (0, s_3, 0, y_{23})$ is locally asymptotically stable in $\Re_+^4$ if and only if:

$$1 - h_1 p < \frac{(h_7 + \tau_2 h_4)}{h_3}, \ s < \frac{h_5 h_7}{(h_2 h_7 - h_3 h_6)} \ \text{and} \ h_2 h_7 > h_3 h_6 \quad (12)$$

PROOF: Consider the function

$$V^{[3]} = p + \frac{1}{e}\left(s - s_3 - s_3 \ln\frac{s}{s_3}\right) + \frac{y_1}{e} + \frac{1}{e}\left(y_2 - y_{23} - y_{23}\ln\frac{y_2}{y_{23}}\right)$$

Clearly, $V^{[3]}: \Re_+^4 \to \Re$ and $V^{[3]}(E_3) = 0$ with $V^{[3]}(E) \neq 0 \ \forall E \neq E_3, E \in \Re_+^4$. Hence it is positive definite function in $\Re_+^4$. Also, the derivative of $V^{[3]}$ with respect to the time t is given as follows.

$$\frac{dV^{[3]}}{dt} = p\left(1 - h_1 p - \frac{h_7}{h_3} - \frac{\tau_2 h_4}{h_3}\right) + \frac{y_1}{e}\left(\frac{h_2 h_7}{h_3} - h_6 - \frac{h_5 h_7}{h_3 s}\right)$$

Hence, $\frac{dV^{[3]}}{dt} < 0$ on subregion of $\Re_+^4$ under the sufficient condition (12), then $V^{[3]}$ is a Lyapunov function on that subregion of $\Re_+^4$ which satisfies condition (12). Therefore E_3 is a locally asymptotically stable but not globally. ■

THEOREM (5): the second disease and prey free equilibrium point $E_4 = (0, s_4, y_{14}, 0)$ is locally asymptotically stable in $\Re_+^4$ if and only if:

$$h_2 h_6 (1 - h_1 p) < (h_5 + h_6)(h_6 + \tau_1 h_4), \ y_1 < y_{14}, \quad s_4 < \frac{h_7}{h_3}$$
$$\text{and} \quad \frac{h_6}{h_4} y_1 < s < s_4 , \quad (13)$$

PROOF: Consider the function

$$V^{[4]} = p + \frac{1}{e}\left(s - s_4 - s_4 \ln\frac{s}{p s_4}\right) + \frac{1}{e}\left(y_1 - y_{14} - y_{14}\ln\frac{y_1}{y_{14}}\right) + \frac{1}{e} y_2$$

Clearly, $V^{[4]}: \Re_+^4 \to \Re$ and $V^{[4]}(E_4) = 0$ with $V^{[4]}(E) \neq 0 \ \forall E \neq E_4, E \in \Re_+^4$. Hence it is positive definite function in $\Re_+^4$. Also, the derivative of $V^{[4]}$ with respect to the time t is given as follows.

$$\frac{dV^{[4]}}{dt} = p(1 - h_1 p - s_4 - \tau_1 y_{14}) + \frac{h_4}{e}(s - s_4) + \frac{h_6}{e}(y_1 - y_{14})$$
$$+ \frac{y_2}{e}(h_3 s_4 - h_7) + \frac{s_4 y_1}{es}(h_2 s - h_5) + \frac{y_{14}}{e}(h_5 - h_2 s)$$

Hence, $\frac{dV^{[4]}}{dt} < 0$ on subregion of $\Re_+^4$ under the sufficient condition (13), then $V^{[4]}$ is a Lyapunov function on that subregion of $\Re_+^4$ which satisfies condition (13). Therefore E_4 is a locally asymptotically stable but not globally. ■

THEOREM (6): the non-hyperbolic equilibrium point $E_5 = (p_5, 0, 0, y_{25})$ is locally asymptotically stable in $\Re_+^4$ if and only if:

$$h_3 y_{25} < h_4 , \ p < p_5 < \min\left\{\frac{h_6}{e\tau_1}, \frac{(h_3 y_{25} - h_4)}{e}\right\} \quad (14)$$
$$\text{and} \quad 1 - h_1 p < \tau_2 y_{25}$$

PROOF: Consider the function

$$V^{[5]} = \left(p - p_5 - p_5 \ln\frac{p}{p_5}\right) + \frac{s}{e} + \frac{y_1}{e} + \frac{1}{e}\left(y_2 - y_{25} - y_{25}\ln\frac{y_2}{y_{25}}\right)$$

Clearly, $V^{[5]}: \Re_+^4 \to \Re$ and $V^{[5]}(E_5) = 0$ with $V^{[5]}(E) \neq 0 \ \forall E \neq E_5, E \in \Re_+^4$. Hence it is positive definite function in $\Re_+^4$. Also, the derivative of $V^{[5]}$ with respect to the time t is given as follows.

$$\frac{dV^{[5]}}{dt} = p(1 - h_1 p - \tau_2 y_{25}) + s\left(\frac{h_4}{e} + p_5 - \frac{h_3 y_{25}}{e}\right) + y_1\left(\tau_1 h_5 - \frac{h_6}{e}\right)$$
$$+ h_1 p_5 (p - p_5)$$

Hence, $\frac{dV^{[5]}}{dt} < 0$ on subregion of $\Re_+^4$ under the sufficient condition (14), then $V^{[5]}$ is a Lyapunov function on that subregion of $\Re_+^4$ which satisfies condition (14). Therefore E_5 is a locally asymptotically stable but not globally. ■

THEOREM (7): the disease free equilibrium point $E_6 = (p_6, s_6, 0, 0)$ is locally asymptotically stable in $\Re_+^4$ if and only if:

$$p < p_6 \ , \ s_6 < \min\left\{\tau_1 y_1, \frac{h_7}{h_3}\right\} \ \text{and} \ s < \frac{h_5}{h_2} \quad (15)$$

PROOF: Consider the function

$$V^{[6]} = \left(p - p_6 - p_6 \ln\frac{p}{p_6}\right) + \frac{1}{e}\left(s - s_6 - s_6 \ln\frac{s}{s_6}\right) + \frac{y_1}{e} + \frac{y_2}{e}$$

Clearly, $V^{[6]}: \Re_+^4 \to \Re$ and $V^{[6]}(E_6) = 0$ with $V^{[6]}(E) \neq 0 \ \forall E \neq E_6, E \in \Re_+^4$. Hence it is positive definite function in $\Re_+^4$. Also, the derivative of $V^{[6]}$ with respect to the time t is given as follows.

$$\frac{dV^{[6]}}{dt} = (p - p_6)(1 - h_1 p) + p_6(s_6 - \tau_1 y_{16}) + \frac{s_6 y_1}{es}(h_2 s - h_5)$$
$$+ \frac{y_2}{e}(h_3 s_6 - h_7) - \left[\frac{h_6}{e} y_1 + p s_6 + 2 s p_6 + \tau_2 p_6 y_2\right]$$

Hence, $\frac{dV^{[6]}}{dt}<0$ on subregion of $\Re_+^4$ under the sufficient condition (15), then $V^{[6]}$ is a Lyapunov function on that subregion of $\Re_+^4$ which satisfies condition (15). Therefore E_6 is a locally asymptotically stable but not globally. ■

THEOREM (8): the equilibrium point $E_7=(p_7,s_7,y_{17},0)$ is locally asymptotically stable in $\Re_+^4$ if and only if:

$$e\tau_2 p_7+h_3 s_7<h_7\ ,\ s<\frac{h_5 s_7 y_1}{(h_5+h_6)y_{17}+(h_2 y_1-h_4)s_7}\ \text{and}\ p<p_7 \qquad (16)$$

PROOF: Consider the function

$$V^{[7]}=\left(p-p_7-p_7\ln\frac{p}{p_7}\right)+\frac{1}{e}\left(s-s_7-s_7\ln\frac{s}{s_7}\right)+\frac{1}{e}\left(y_1-y_{17}-y_{17}\ln\frac{y_1}{y_{17}}\right)+\frac{y_2}{e}$$

Clearly, $V^{[7]}:\Re_+^4\to\Re$ and $V^{[7]}(E_7)=0$ with $V^{[7]}(E)\neq 0\ \ \forall E\neq E_7\,,E\in\Re_+^4$. Hence it is positive definite function in $\Re_+^4$. Also, the derivative of $V^{[7]}$ with respect to the time t is given as follows.

$$\frac{dV^{[7]}}{dt}=(p-p_7)(1-h_1p)+s\left(p_7+\frac{h_4}{e}-\frac{h_2 y_{17}}{e}\right)-p(s_7+\tau_1 y_{17})+y_1\left(\tau_1 p_7-\frac{h_6}{e}\right)+y_2\left(\tau_2 p_7+\frac{h_3}{e}s_7-\frac{h_7}{e}\right)+\left((h_5+h_6)\frac{y_{17}}{e}+(h_2 y_1-h_4)\frac{s_7}{e}-\frac{h_5}{es}s_7 y_1\right)$$

Hence, $\frac{dV^{[7]}}{dt}<0$ on subregion of $\Re_+^4$ under the sufficient condition (16), then $V^{[7]}$ is a Lyapunov function on that subregion of $\Re_+^4$ which satisfies condition (16). Therefore E_7 is a locally asymptotically stable but not globally. ■

THEOREM (9): the equilibrium point $E_8=(p_8,s_8,0,y_{28})$ is locally asymptotically stable in $\Re_+^4$ if and only if:

$$e\tau_1 p_8+h_2 s_8<h_6\ ,\ esp_8(h_7+\tau_2 h_4)<h_5(h_7-e\tau_2 p_8)\ \text{and}\ p<p_8 \qquad (17)$$

PROOF: Consider the function

$$V^{[8]}=\left(p-p_8-p_8\ln\frac{p}{p_8}\right)+\frac{1}{e}\left(s-s_8-s_8\ln\frac{s}{s_8}\right)+\frac{y_1}{e}+\frac{1}{e}\left(y_2-y_{28}-y_{28}\ln\frac{y_2}{y_{28}}\right)$$

Clearly, $V^{[8]}:\Re_+^4\to\Re$ and $V^{[8]}(E_8)=0$ with $V^{[8]}(E)\neq 0\ \ \forall E\neq E_8\,,E\in\Re_+^4$. Hence it is positive definite function in $\Re_+^4$. Also, the derivative of $V^{[8]}$ with respect to the time t is given as follows.

$$\frac{dV^{[8]}}{dt}=(p-p_8)(1-h_1p)-p(s_8+\tau_2 y_{28})+y_1\left(\tau_1 p_8+\frac{h_2}{e}s_8-\frac{h_6}{e}\right)+\left(s(h_7 y_{28}-h_4 s_8)-h_5 s_8 y_1\right)$$

Hence, $\frac{dV^{[8]}}{dt}<0$ on subregion of $\Re_+^4$ under the sufficient condition (17), then $V^{[8]}$ is a Lyapunov function on that subregion of $\Re_+^4$ which satisfies condition (17). Therefore E_8 is a locally asymptotically stable but not globally. ■

THEOREM (10): the equilibrium point $E_9=(0,s_9,y_{19},y_{28})$ is locally asymptotically stable in $\Re_+^4$ if and only if:

$$1-h_1p<s_9+\tau_1 y_{19}+\tau_2 y_{29}\ ,\ y_{29}>\frac{h_4 h_5}{h_3(h_5+h_6)}\quad and\quad \frac{h_6 h_7 y_{29}}{h_5(h_4-h_3 y_{29})}<s<s_9 \qquad (18)$$

PROOF: Consider the function

$$V^{[9]}=p+\frac{1}{e}\left(s-s_9-s_9\ln\frac{s}{s_9}\right)+\frac{1}{e}\left(y_1-y_{19}-y_{19}\ln\frac{y_1}{y_{19}}\right)+\frac{1}{e}\left(y_2-y_{29}-y_{29}\ln\frac{y_2}{y_{29}}\right)$$

Clearly, $V^{[9]}:\Re_+^4\to\Re$ and $V^{[9]}(E_9)=0$ with $V^{[9]}(E)\neq 0\ \ \forall E\neq E_9\,,E\in\Re_+^4$. Hence it is positive definite function in $\Re_+^4$. Also, the derivative of $V^{[9]}$ with respect to the time t is given as follows.

$$\frac{dV^{[9]}}{dt}=p(1-h_1p-s_9-\tau_1 y_{19}-\tau_2 y_{29})+\frac{1}{e}\left[\frac{h_5 s}{h_6}(h_3 y_{29}-h_4)+h_7 y_{29}\right]+\frac{h_5 y_1}{es}(s-s_9)+\frac{(h_5+h_6)}{eh_2h_6}\left(h_4 h_5-h_3 y_{29}(h_5+h_6)\right)$$

Since E_9 exists if and only if $h_4>h_3 y_{29}$, in addition condition(18) guarantee that $\frac{dV^{[9]}}{dt}<0$ on subregion of $\Re_+^4$, then $V^{[9]}$ is a Lyapunov function on that subregion which satisfies condition (18). Therefore E_9 is a locally asymptotically stable but not globally. ■

THEOREM (11): The coexistence equilibrium point E_{10} is locally asymptotically stable in $\Re_+^4$ if and only if:

$$\max\left\{\frac{h_6}{e\tau_1},\frac{-h_4}{e},\frac{h_7}{e\tau_2},\right\}<p<p_{10}\ ,\ s+\tau_1 y_1+\tau_2 y_2<1-h_1 p\ ,$$
$$y_2<\min\left\{y_{210}\,,\frac{sy_{210}}{s_{10}}\right\}s<\min\left\{s_{10}\,,\frac{h_5}{h_2}\right\} \qquad (19)$$
$$and\quad \frac{sy_{110}}{s_{10}}<y_1<y_{110}$$

PROOF: Consider the function

$$V^{[10]}=\left(s-p_{10}-p_{10}\ln\frac{p}{p_{10}}\right)+\frac{1}{e}\left(s-s_{10}-s_{10}\ln\frac{s}{s_{10}}\right)+\frac{1}{e}\left(y_1-y_{110}-y_{110}\ln\frac{y_1}{y_{110}}\right)+\frac{1}{e}\left(y_2-y_{210}-y_{210}\ln\frac{y_2}{y_{210}}\right)$$

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Clearly, $V^{[10]}:\Re_+^4 \to \Re$ and $V^{[10]}(E_{10})=0$ with $V^{[10]}(E)\neq 0 \quad \forall\, E \neq E_{10}\,,\, E\in\Re_+^4$. Hence it is positive definite function in $\Re_+^4$. Also, the derivative of $V^{[10]}$ with respect to the time t is given as follows.

$$\frac{dV^{[10]}}{dt}=(p_{10}-p)\left[-(1-h_1p)+s+\tau_1 y_1+\tau_2 y_2\right]+(ep+h_4)(s-s_{10})$$
$$+(e\tau_2 p-h_7)(y_2-y_{210})+(e\tau_1 p-h_6)(y_1-y_{110})$$
$$+\frac{1}{s}(s_{10}y_1-sy_{110})(h_2 s-h_5)+h_3(s_{10}y_2-sy_{210})$$

Since E_{10} exists if and only if $h_4>h_3 y_{29}$, in addition condition(19) guarantee that $\frac{dV^{[10]}}{dt}<0$ on subregion of $\Re_+^4$, then $V^{[10]}$ is a Lyapunov function on that subregion which satisfies condition (19). Therefore E_{10} is a locally asymptotically stable but not globally. ■

5. Numerical simulations:

We give some numerical analysis in support our theoretical findings. The system (3) is solved numerically, for different sets of parameters, using predictor-corrector method with six order Runge-Kutta method, and then the time series for the trajectories of system (3) are draw. Now before we go farther with numerical analysis, We will use the solid line (——) for p, dash line (— —) for s, dot line (….) for y_1, dash-dot line (— . —) for y_2 and the initial point $(0.75, 0.75, 0.75, 0.75)$. in the all of the following figures.

Now to show the stable of axial equilibrium point on the s-axis E_1 used the following set of hypothetical parameters values:

$$h_1=50\,,\,h_2=0.01\,,\,h_3=0.01\,,\,h_4=0\,,\,h_5=0.08\,,$$
$$h_6=0.1\,,\,h_7=0.5\,,\,e=0.4\,,\,\tau_1=0.3\,,\,\tau_2=0.1 \qquad (20)$$

In Fig.(1), the system (3) approaches asymptotically to the equilibrium point $E_1=(0, 1.059, 0, 0)$.

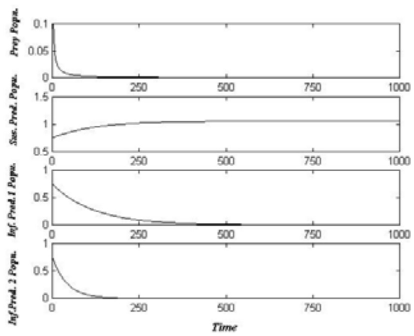


Fig.(1): time series of the trajectories of the system (3) which shows E_1 is a locally asymptotically stable point.

Now to show the stable of axial equilibrium point on the p-axis equilibrium point E_2 used the following set of hypothetical parameters values:

$$h_1=50\,,\,h_2=0.01\,,\,h_3=0.01\,,\,h_4=-0.04\,,\,h_5=0.08\,,$$
$$h_6=0.1\,,\,h_7=0.5\,,\,e=0.4\,,\,\tau_1=0.3\,,\,\tau_2=0.1 \qquad (21)$$

the system (3) approaches asymptotically to the equilibrium point $E_2=(0.02, 0, 0, 0)$ as show as in Fig.(2).

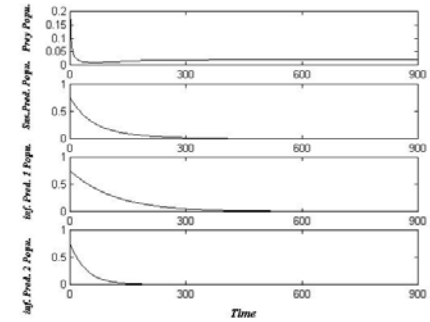


Fig.(2): time series of the trajectories of the system (3) which shows E_2 is a globally asymptotically stable point.

Now to show the stable of first disease and prey free equilibrium point E_3 used the following set of hypothetical parameters values:

$$h_1=50\,,\,h_2=0.8\,,\,h_3=0.1\,,\,h_4=0.2\,,\,h_5=0.2\,,$$
$$h_6=0.08\,,\,h_7=0.03\,,\,e=0.8\,,\,\tau_1=0.3\,,\,\tau_2=0.9 \qquad (22)$$

In Fig.(3), the system (3) approaches asymptotically to the equilibrium point $E_3=(0, 0.295, 0, 2.018)$.

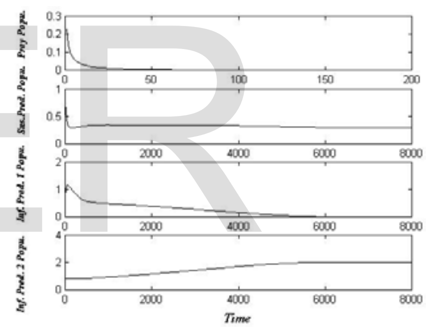
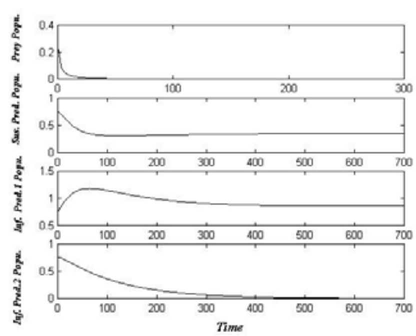


Fig.(3): time series of the trajectories of the system (3) which shows E_3 is a locally asymptotically stable point.

Now to show the stable of second disease and prey free equilibrium point E_4 used the following set of hypothetical parameters values:

$$h_1=50\,,\,h_2=0.8\,,\,h_3=0.1\,,\,h_4=0.2\,,\,h_5=0.2\,,$$
$$h_6=0.08\,,\,h_7=0.2\,,\,e=0.8\,,\,\tau_1=0.8\,,\,\tau_2=0.9 \qquad (23)$$

In Fig.(4), the system (3) approaches asymptotically to the stable equilibrium point $E_4=(0, 0.35, 0.875, 0)$



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Now to show the stable of first disease and susceptible predator free equilibrium point E_5 used the following set of hypothetical parameters values:

$$h_1 = 50 \, , h_2 = 0.8 \, , h_3 = 0.1 \, , h_4 = -0.01 \, , h_5 = 0.2 \, , \\ h_6 = 0.08 \, , h_7 = 0.01 \, , e = 0.8 \, , \tau_1 = 0.8 \, , \tau_2 = 0.9 \quad (24)$$

In Fig.(5), the system (3) approaches asymptotically to the stable equilibrium point $E_5 = (0.014, 0, 0, 0.385)$

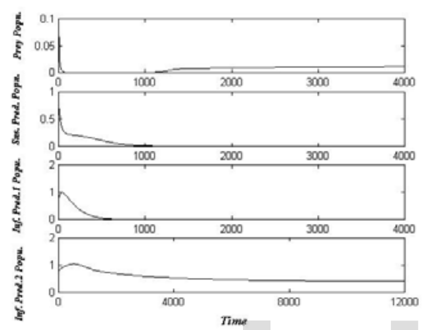


Fig.(5): time series of the trajectories of the system (3) which shows E_5 is a locally asymptotically stable point.

Now to show the stable of disease free equilibrium point E_6 used the following set of hypothetical parameters values:

$$h_1 = 50 \, , h_2 = 0.1 \, , h_3 = 0.2 \, , h_4 = -0.001 \, , h_5 = 0.08 \, , \\ h_6 = 0.1 \, , h_7 = 0.5 \, , e = 0.4 \, , \tau_1 = 0.3 \, , \tau_2 = 0.1 \quad (25)$$

In Fig.(6), the system (3) approaches asymptotically to the stable equilibrium point $E_6 = (0.01, 0.505, 0, 0)$

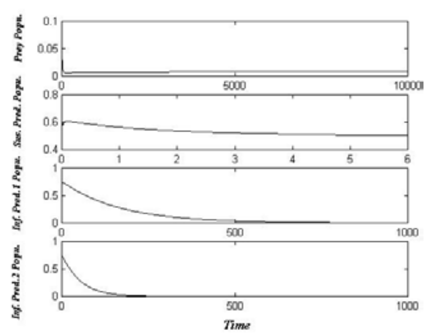
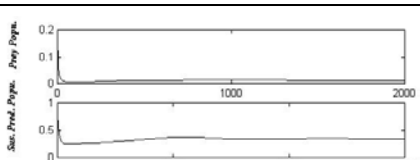


Fig.(6): time series of the trajectories of the system (3) which shows E_6 is a locally asymptotically stable point.

Now to show the stable of second disease free equilibrium point E_7 used the following set of hypothetical parameters values:

$$h_1 = 50 \, , h_2 = 0.8 \, , h_3 = 0.01 \, , h_4 = -0.0001 \, , h_5 = 0.2 \, , \\ h_6 = 0.08 \, , h_7 = 0.5 \, , e = 0.5 \, , \tau_1 = 0.4 \, , \tau_2 = 0.8 \quad (26)$$

As shown as in Fig.(7), the system (3) approaches asymptotically to the stable equilibrium point $E_7 = (0.013, 0.344, 0.046, 0)$.



Now to show the stable of first disease free equilibrium point E_8 used the following set of hypothetical parameters values:

$$h_1 = 50 \, , h_2 = 0.5 \, , h_3 = 0.3 \, , h_4 = 0.1 \, , h_5 = 0.1 \, , \\ h_6 = 0.2 \, , h_7 = 0.1 \, , e = 0.1 \, , \tau_1 = 0.1 \, , \tau_2 = 0.2 \quad (27)$$

As shown as in Fig.(8),the system (3) approaches asymptotically to the equilibrium point $E_8 = (0.013, 0.332, 0, 0.342)$.

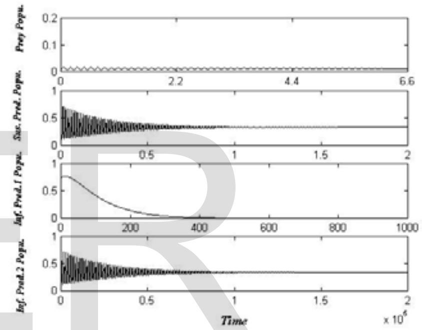


Fig.(8): time series of the trajectories of the system (3) which shows E_8 is a locally asymptotically stable point.

Now to show the stable of prey free equilibrium point E_9 used the following set of hypothetical parameters values:

$$h_1 = 50 \, , h_2 = 0.01 \, , h_3 = 0.05 \, , h_4 = 0.45 \, , h_5 = 0.06 \, , \\ h_6 = 0.03 \, , h_7 = 0.45 \, , e = 0.4 \, , \tau_1 = 0.3 \, , \tau_2 = 0.3 \quad (28)$$

As shown as in Fig.(9), the system (3) approaches asymptotically to the equilibrium point $E_9 = (0, 8.98, 0.555, 0.16)$.

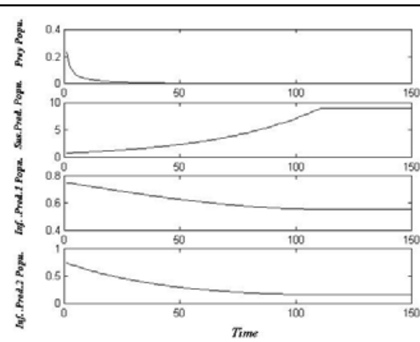


Fig.(9): time series of the trajectories of the system (3) which shows E_9 is a locally asymptotically stable point.

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Finally, to understand of dynamical behavior at the coexistence equilibrium point E_{10} the following set of hypothetical parameter values is chosen:

$$h_1 = 48 , h_2 = 0.88 , h_3 = 0.1 , h_4 = -0.00019 , h_5 = 0.2 , h_6 = 0.08 , h_7 = 0.037 , e = 0.7 , \tau_1 = 0.5 , \tau_2 = 0.9 \quad (29)$$

As shown as in Fig.(10), the system(3) approaches asymptotically to the stable equilibrium point $E_{10} = (0.013, 0.309, 0.018, 0.077)$.

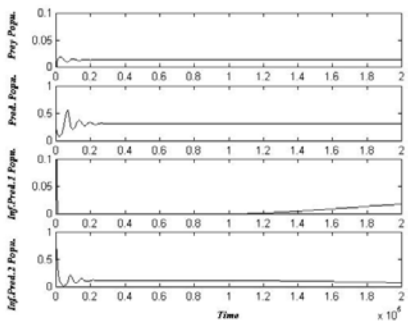

Fig.(10): time series of the trajectories of the system (3) which shows E_{10} is a locally asymptotically stable point.

6. CONCLUSIONS AND DISCUSSION:

The stability of model has been studied with linear functional response and numerical response. We propose only one model contain more than one model as following:

First, we investigated that the vanishing equilibrium point $E_0 = (0, 0, 0, 0)$ is always unstable, the conditions (10) for which the axial equilibrium point on the s-axis $E_1 = (0, 1.059, 0, 0)$ is locally asymptotically stable but not globally, and axial equilibrium point on the p-axis $E_2 = (0.02, 0, 0, 0)$ is locally asymptotically stable also it's globally.

Second, we have SI- epidemic model with the equilibrium point E_3, and show that $E_3 = (0, 0.295, 0, 2.018)$ is locally asymptotically stable but not globally with conditions (12).

Third, we have SIS- epidemic model with the equilibrium point E_4, and show that, in theorem (5), $E_4 = (0, 0.35, 0.875, 0)$ is locally asymptotically stable but not globally.

Fourth, we have prey-infected predator by SI model with the equilibrium point E_5, and show that, in theorem (6), $E_5 = (0.014, 0, 0, 0.385)$ is locally asymptotically stable but not globally.

Fifth, we have prey-predator model with the equilibrium point E_6, and show that, in theorem (7), $E_6 = (0.01, 0.505, 0, 0)$ is locally asymptotically stable but not globally.

Sixth, we have prey-predator model with SIS-disease in predator, investigated the condition (16) for which the equilibrium point E_7 is stable, and numerically show that $E_7 = (0.013, 0.344, 0.046, 0)$ is locally asymptotically stable but not globally.

Seventh, we have prey-predator model with SI-disease in predator, investigated the condition (17) for which the equilibrium point E_8 is stable, and numerically show that $E_8 = (0.013, 0.332, 0, 0.342)$ is locally asymptotically stable but not globally.

Eighth, we have epidemic model spread two diseases the population, and investigated in the theorem (10) the equilibrium point E_9 is stable, and numerically show that $E_9 = (0, 8.98, 0.555, 0.16)$ is locally asymptotically stable but not globally.

Finally, we investigated the condition (19) for which the coexistence equilibrium point E_{10} is stable, more than, numerically prove that $E_{10} = (0.013, 0.309, 0.018, 0.077)$ is locally asymptotically stable but not globally. In general, use the Lyapunov function to find the stability of the system (3) at each most of its equilibrium points.

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